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A revisit of a cylindrically anisotropic tube subjected to pressuring, shearing, torsion, extension and a uniform temperature change

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Abstract

The problem of a cylindrically anisotropic tube or bar was seemed to be first examined by Lekhnitskii (1981) [Lekhnitskii, S.G., 1981. Theory of Elasticity of an Anisotropic Body. (Trans. from the revised 1977 Russian edition.) Mir, Moscow]. Recently, a thorough investigation of the subject was performed by Ting (1996) [Ting, T.C.T., 1996. Pressuring, shearing, torsion and extension of a circular tube or bar of cylindrically anisotropic material. Proc. Roy. Soc. Lond. A452, 2397–2421] in which a formulation akin to that of Stroh's formalism is employed to resolve the boundary value problem subjected to a uniform pressure, shearing, torsion and uniform extension. In a continuing paper, Ting (1999) [Ting, T.C.T., 1999. New solutions to pressuring, shearing, torsion and extension of a cylindrically anisotropic elastic circular tube or bar. Proc. Roy. Soc. Lond, to appear.] rederived the solutions based on a modified formalism of Lekhnitskii, in which the solutions are in terms of elastic compliances, reduced elastic compliances as well as doubly reduced compliance. The results are much more compact and simpler than those of the earlier one. Independently, in this work, we construct the governing system also under the Lekhnitskii's framework. Nevertheless, the present work and Ting's formulation (1999) are not alike. Besides the loads considered in Ting (1996, 1999), we add the effect of a uniform temperature change in the formulation. The assumption that *the stresses depend only on r* makes it possible to incorporate the various loading cases considered. In addition to the explicit forms of admissible stresses, we derive the admissible displacements which are ensured to be single-valued for a multiply-connected domain. In contrast to the Ting's works (1996, 1999), which often require superpositions of two or more basic solutions, the present solutions offer complete forms of solutions ready for direct calculations. We also report that, as in rectilinearly anisotropic solids, an entire analogy is observed between the fields of a uniform axial extension and a uniform temperature change in cylindrically anisotropic solids.

Keywords: Cylindrical anisotropy; Lekhnitskii's formalism; Anisotropic elasticity; Circular tube

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1. Introduction

Some years ago, Ting (1996) presented a thorough investigation of a circular tube or bar of cylindrically anisotropic material subjected to pressuring, shearing, torsion and extension (see Figs. 1 and 2). Stroh formalism is devised for cylindrically anisotropic materials, in which the fields are expressed in terms of elastic moduli. The derivation assumes that the displacements only depend on the radial distance r. Despite the fact that the formulation has the merits of simplicity and clarity, the solutions are rather cumbersome, and in certain situations they may not be very convenient for quick references.

Exactly the same problem was seemed to be first examined by Lekhnitskii (1981), in which the same types of loads *except the shearing loadings* are considered. In that study, a general framework of generalized plane deformations (stresses do not depend on the axial direction z) is first established. Specific results, which are in terms of elastic compliances, are then derived for the considered loadings under the assumption that *the characterized stress potentials F and Ψ are only functions of r*. This assumption indeed excludes the possibility of incorporating the shearing loads in the formulation.

The present work is mainly motivated by the two previous works with the objective to offer simple exact solutions for applications. Besides the loading conditions considered by Ting (1996), we add the effect of a uniform temperature change in the formulation. Lekhnitskii's formalism is adopted under the condition that *the stresses, instead of the stress potentials, depend only on r*. This makes it possible to incorporate the various loading cases considered. We mention that Lekhnitskii's assumption, i.e. $F = F(r)$, $\Psi = \Psi(r)$, is more restricted than the present one, $\boldsymbol{\sigma} = \boldsymbol{\sigma}(r)$. Note that $F(r)$ and $\Psi(r)$ always imply the stresses are functions of r, but not conversely. In addition to the explicit forms of admissible stresses, we derive the admissible displacements which are ensured to be single-valued for a multiply-connected domain. In contrast to Ting's (1996) work, and a continuing paper (Ting, 1999) which often require superpositions of two or more basic solutions, the present solutions offer complete forms of

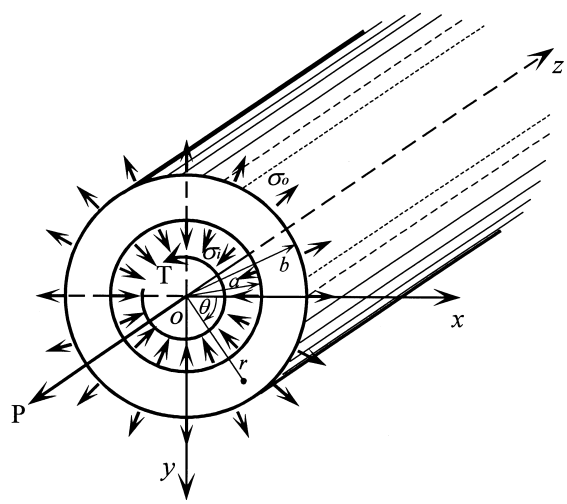

Fig. 1. A schematic representation of a circular tube subjected to radial traction, torsion, extension.

solutions ready for numerical calculations. We also find an interesting analogy between the fields of a uniform axial extension and a uniform temperature change for a cylindrically anisotropic body. This phenomenon was also observed in rectilinearly anisotropic solids (Chen, 1998).

Just prior to the completion of this work, Professor Ting informed the leading author that he had rederived his earlier results (1996) based on a modification of Lekhnitskii formalism. The new results (Ting, 1999), which are written in terms of elastic compliance, reduced elastic compliances as well as the doubly reduced elastic compliances, have much simpler forms compared with those of the earlier one (Ting, 1996). Although the present study and Ting's (1999) work both employ modified Lekhnitskii's formalism, the formulation is *not* alike.

Relevant earlier results include Avery and Herakovich (1986), Chen et al. (1990), Hashin (1990) and Christensen (1994), in which they are mainly concerned with the cylindrically orthotropic solid. To begin with, let us first briefly describe the constitutive relations of cylindrically anisotropic solids. Materials with cylindrical anisotropy possess constant properties in a cylindrical coordinate system (r, ϕ, z), namely properties in tangential, radial, and axial directions are different from each other. They are not uncommon in nature. For example, they appear in carbon fiber (Dresselhaus et al., 1988) and in tree trunks. Anisotropy of this kind is characterized by the fact that the material properties at any point along the r direction are exactly the same; also all directions parallel to the ϕ and z directions are constant. The constitutive equations can be written in the form

$$\boldsymbol{\varepsilon} = \mathscr{S}\boldsymbol{\sigma} + \boldsymbol{\alpha}\Delta T \tag{1}$$

where $\mathscr{S} = (s_{ij})$, $i, j = 1$–6, are the elastic compliance which link the stresses $\boldsymbol{\sigma}$ and the strains $\boldsymbol{\varepsilon}$, $\boldsymbol{\alpha} = (\alpha_i)$ are the thermal expansion coefficients and ΔT is the uniform temperature change. The stresses $\boldsymbol{\sigma}$ and strains $\boldsymbol{\varepsilon}$ are written on the short notations

$$\boldsymbol{\sigma} = \left[\sigma_r, \sigma_\phi, \sigma_z, \sigma_{\phi z}, \sigma_{rz}, \sigma_{r\phi}\right]^{\mathrm{T}},$$

$$\boldsymbol{\varepsilon} = \left[\varepsilon_r, \varepsilon_\phi, \varepsilon_z, 2\varepsilon_{\phi z}, 2\varepsilon_{rz}, 2\varepsilon_{r\phi}\right]^{\mathrm{T}}. \tag{2}$$

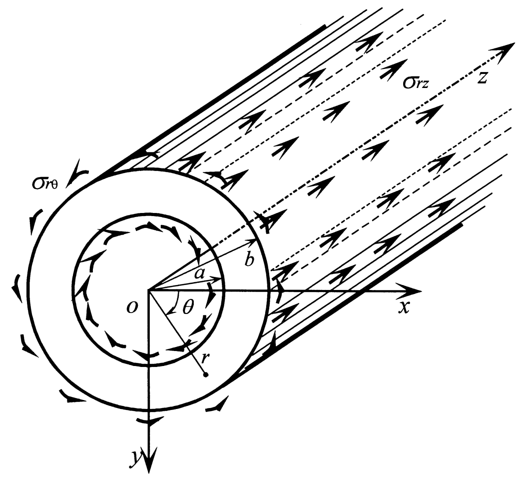

Fig. 2. A schematic representation of a circular tube subjected to in-plane and anti-plane shears.

2. General framework

Consider a cylindrically anisotropic body of cylindrical shape in which all cross sections are of the same geometry along the axial direction. Suppose the body is subjected to boundary conditions over the lateral surface and at the end sections so that the stresses depend on r only, but not on ϕ or z. For example, a circular tube subjected to radial forces, shearing stresses, or subjected to an axial extension or a twisting moment at its end sections is an example of this. Under the prerequisite $\sigma_i = \sigma_i(r)$, equilibrium equations are (cf. Ting, 1999)

$$\frac{\mathrm{d}\sigma_r}{\mathrm{d}r} + \frac{\sigma_r - \sigma_\phi}{r} = 0, \quad \frac{\mathrm{d}\sigma_{r\phi}}{\mathrm{d}r} + 2\frac{\sigma_{r\phi}}{r} = 0, \quad \frac{\mathrm{d}\sigma_{rz}}{\mathrm{d}r} + \frac{\sigma_{rz}}{r} = 0. \tag{3}$$

The latter two equations readily provide

$$\sigma_{r\phi} = \frac{\sigma^0_{r\phi}}{r^2}, \qquad \sigma_{rz} = \frac{\sigma^0_{rz}}{r}, \tag{4}$$

in which $\sigma^0_{r\phi}$, and σ^0_{rz} are certain constant quantities to be determined from boundary conditions (see also Ting, 1996).

Since the stresses depend on r, so do the strains. The compatibility conditions can be simplified as

$$\frac{\mathrm{d}^2\varepsilon_z}{\mathrm{d}r^2} = 0, \quad \frac{1}{r}\frac{\mathrm{d}\varepsilon_z}{\mathrm{d}r} = 0, \quad \frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\varepsilon_\phi}{\mathrm{d}r}\right) - r\frac{\mathrm{d}\varepsilon_r}{\mathrm{d}r} = 0, \quad \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r\varepsilon_{\phi z})\right) = 0. \tag{5}$$

The first two equations suggest that ε_z must be a constant, say ε^0_z, and the remaining ones are recast as

$$r\frac{\mathrm{d}^2\varepsilon_\phi}{\mathrm{d}r^2} + 2\frac{\mathrm{d}\varepsilon_\phi}{\mathrm{d}r} - \frac{\mathrm{d}\varepsilon_r}{\mathrm{d}r} = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}r}(r\varepsilon_{\phi z}) = \vartheta r, \tag{6}$$

where ϑ is an integration constant, but physically it corresponds to a twisting angle per unit length along the axial direction. For later convenience, let us rearrange Eq. (1) as

$$\begin{Bmatrix} \varepsilon_r \\ \varepsilon_\phi \\ 2\varepsilon_{\phi z} \\ 2\varepsilon_{rz} \\ 2\varepsilon_{r\phi} \end{Bmatrix} = \begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{14} & \beta_{15} & \beta_{16} \\ \beta_{12} & \beta_{22} & \beta_{24} & \beta_{25} & \beta_{26} \\ \beta_{14} & \beta_{24} & \beta_{44} & \beta_{45} & \beta_{46} \\ \beta_{15} & \beta_{25} & \beta_{45} & \beta_{55} & \beta_{56} \\ \beta_{16} & \beta_{26} & \beta_{36} & \beta_{46} & \beta_{66} \end{bmatrix} \begin{Bmatrix} \sigma_r \\ \sigma_\phi \\ \sigma_{\phi z} \\ \sigma_{rz} \\ \sigma_{r\phi} \end{Bmatrix} + \begin{Bmatrix} s'_{13} \\ s'_{23} \\ s'_{34} \\ s'_{35} \\ s'_{36} \end{Bmatrix} \varepsilon^0_z + \begin{Bmatrix} \alpha'_1 \\ \alpha'_2 \\ \alpha'_4 \\ \alpha'_5 \\ \alpha'_6 \end{Bmatrix} \Delta T, \tag{7}$$

$$-\sigma_z = s'_{13}\sigma_r + s'_{23}\sigma_\phi + s'_{34}\sigma_{\phi z} + s'_{35}\sigma_{rz} + s'_{36}\sigma_{r\phi} - E_z\varepsilon^0_z + E_z\alpha_3\Delta T, \tag{8}$$

where

$$\beta_{ij} = s_{ij} - \frac{s_{i3}s_{j3}}{s_{33}}, \quad s'_{i3} = \frac{s_{i3}}{s_{33}}, \quad \alpha'_i = \alpha_i - \frac{\alpha_3 s_{i3}}{s_{33}}, \quad i,j = 1,2,4,5,6 \tag{9}$$

and $E_z = 1/s_{33}$ is the axial Young's modulus.

To solve the system (3), (6), (7) and (8), we introduce the stress function F governed by

$$\sigma_r = \frac{1}{r}\frac{\mathrm{d}F}{\mathrm{d}r}, \qquad \sigma_\phi = \frac{\mathrm{d}^2F}{\mathrm{d}r^2}, \tag{10}$$

so that Eq. (3a) is fulfilled and, for equidimensional purpose, let

$$\sigma_{\phi z} = -\frac{\mathrm{d}^2\Psi}{\mathrm{d}r^2}. \tag{11}$$

Upon substituting Eqs. (10) and (11) into Eq. (6), the solution fields are governed by

$$\begin{cases} \beta_{22}\left(\dfrac{\mathrm{d}^4F}{\mathrm{d}r^4} + \dfrac{2}{r}\dfrac{\mathrm{d}^3F}{\mathrm{d}r^3}\right) + \beta_{11}\left(-\dfrac{1}{r^2}\dfrac{\mathrm{d}^2F}{\mathrm{d}r^2} + \dfrac{1}{r^3}\dfrac{\mathrm{d}F}{\mathrm{d}r}\right) - \beta_{24}\dfrac{\mathrm{d}^4\Psi}{\mathrm{d}r^4} + \left(\beta_{14}+2\beta_{24}\right)\dfrac{1}{r}\dfrac{\mathrm{d}^3\Psi}{\mathrm{d}r^3} \\ \quad = \dfrac{-2\left(\beta_{16}+\beta_{26}\right)\sigma^0_{r\phi}}{r^4} - \dfrac{\beta_{15}\sigma^0_{rz}}{r^3}, -\beta_{24}\dfrac{\mathrm{d}^3F}{\mathrm{d}r^3} - \left(\beta_{14}+\beta_{24}\right)\dfrac{1}{r}\dfrac{\mathrm{d}^2F}{\mathrm{d}r^2} + \beta_{44}\left(\dfrac{\mathrm{d}^3\Psi}{\mathrm{d}r^3} + \dfrac{1}{r}\dfrac{\mathrm{d}^2\Psi}{\mathrm{d}r^2}\right) \\ \quad = \dfrac{s'_{34}\varepsilon^0_z}{r} - \dfrac{\beta_{46}\sigma^0_{r\phi}}{r^3} - 2\vartheta. \end{cases} \tag{12}$$

The displacement fields can be directly integrated from Eqs. (7) and (8) in the forms (Lekhnitskii, 1981)

$$u_r = U + \left[z(\omega_2\cos\phi - \omega_1\sin\phi) + u^0_1\cos\phi + u^0_2\sin\phi\right],$$

$$u_\phi = V + \vartheta zr + \left[-z(\omega_2\sin\phi + \omega_1\cos\phi) + \omega_3 r - u^0_1\sin\phi + u^0_2\cos\phi\right],$$

$$u_z = W + \varepsilon^0_z z + \left[r(\omega_1\sin\phi - \omega_2\cos\phi) + u^0_3\right], \tag{13}$$

where ω_1, ω_2, ω_3 are the rigid body rotations about three axes, x_1, x_2, x_3, respectively, and u^0_1, u^0_2, u^0_3 are the rigid body displacements. The function U, V, W are related by

$$\frac{\partial U}{\partial r} = \beta_{11}\sigma_r + \beta_{12}\sigma_\phi + \beta_{14}\sigma_{\phi z} + \beta_{15}\sigma_{rz} + \beta_{16}\sigma_{r\phi} + s'_{13}\varepsilon^0_z + \alpha'_1\Delta T, \tag{14a}$$

$$\frac{1}{r}\frac{\partial V}{\partial \phi} + \frac{U}{r} = \beta_{12}\sigma_r + \beta_{22}\sigma_\phi + \beta_{24}\sigma_{\phi z} + \beta_{25}\sigma_{rz} + \beta_{26}\sigma_{r\phi} + s'_{23}\varepsilon^0_z + \alpha'_2\Delta T, \tag{14b}$$

$$\frac{1}{r}\frac{\partial U}{\partial \phi} + \frac{\partial V}{\partial r} - \frac{V}{r} = \beta_{16}\sigma_r + \beta_{26}\sigma_\phi + \beta_{46}\sigma_{\phi z} + \beta_{56}\sigma_{rz} + \beta_{66}\sigma_{r\phi} + s'_{36}\varepsilon^0_z + \alpha'_6\Delta T, \tag{14c}$$

$$\frac{\partial W}{\partial r} = \beta_{15}\sigma_r + \beta_{25}\sigma_\phi + \beta_{45}\sigma_{\phi z} + \beta_{55}\sigma_{rz} + \beta_{56}\sigma_{r\phi} + s'_{35}\varepsilon^0_z + \alpha'_5\Delta T, \tag{14d}$$

$$\frac{1}{r}\frac{\partial W}{\partial \phi} = \beta_{14}\sigma_r + \beta_{24}\sigma_\phi + \beta_{44}\sigma_{\phi z} + \beta_{45}\sigma_{rz} + \beta_{46}\sigma_{r\phi} + s'_{34}\varepsilon^0_z + \alpha'_4\Delta T - \vartheta r. \tag{14e}$$

3. Admissible fields

The system of differential equation (12) is equidimensional, which can be resolved by introducing a new variable $s = \ln r$. Defining the differential operator $D = \mathrm{d}/\mathrm{d}s$, the system (12) is decoupled as

$$\begin{cases} D^2(D-1)^2(D-2)\left[(D-1)^2-k^2\right]F = 12(\beta_{14}-2\beta_{24})\vartheta e^{3s}/\left(\beta_{22}\beta_{44}-\beta_{24}^2\right), \\ D^2(D-1)^2(D-2)\left[(D-1)^2-k^2\right]\Psi = 6(\beta_{11}-4\beta_{22})\vartheta e^{3s}/\left(\beta_{22}\beta_{44}-\beta_{24}^2\right). \end{cases} \tag{15}$$

The characteristic values of the systems are: 2, $1-k$, $1+k$ together with two double roots 1 and 0, where

$$k = \sqrt{\frac{\beta_{44}\beta_{11}-\beta_{14}^2}{\beta_{44}\beta_{22}-\beta_{24}^2}} \tag{16}$$

Note that the numerator and denominator of Eq. (16) are principal minors of elastic compliance and thus their values are always positive. For all transversely isotropic and isotropic solids, $\beta_{11}=\beta_{22}$, $\beta_{14}=\beta_{24}=0$, it follows that $k=1$. Of course, there are certain anisotropic materials, although unlikely occurs in practice, that $k=1$. Naturally, depending on the values of k, the detailed solutions will take different forms. We summarize in the following cases.

3.1. Anisotropic materials with $k\neq1$ or $k\neq2$

For conciseness of the formulae we define the following short notations

$$\eta_1 = \frac{-\beta_{44}(\beta_{16}+\beta_{26})+\beta_{46}(\beta_{14}+\beta_{24})}{\beta_{11}\beta_{44}-\beta_{14}^2-\left(\beta_{22}\beta_{44}-\beta_{24}^2\right)}, \quad \eta_2 = \frac{-\beta_{46}(\beta_{11}-\beta_{22})+(\beta_{16}+\beta_{26})(\beta_{14}-\beta_{24})}{\beta_{11}\beta_{44}-\beta_{14}^2-\left(\beta_{22}\beta_{44}-\beta_{24}^2\right)}$$

$$\mu_1 = \frac{2\beta_{24}-\beta_{14}}{\beta_{11}\beta_{44}-\beta_{14}^2-4\left(\beta_{22}\beta_{44}-\beta_{24}^2\right)}, \quad \mu_2 = \frac{4\beta_{22}-\beta_{11}}{\beta_{11}\beta_{44}-\beta_{14}^2-4\left(\beta_{22}\beta_{44}-\beta_{24}^2\right)}$$

$$\rho_1 = \frac{\beta_{14}\beta_{45}-\beta_{15}\beta_{44}}{\beta_{11}\beta_{44}-\beta_{14}^2}, \quad \rho_2 = \frac{\beta_{11}\beta_{45}-\beta_{15}\beta_{14}}{\beta_{11}\beta_{44}-\beta_{14}^2},$$

$$\xi_\varepsilon = \frac{s'_{34}(\beta_{14}-\beta_{24})-\beta_{44}(s'_{13}-s'_{23})}{\beta_{11}\beta_{44}-\beta_{14}^2-\left(\beta_{22}\beta_{44}-\beta_{24}^2\right)}, \quad \xi_t = \frac{\alpha'_4(\beta_{14}-\beta_{24})-\beta_{44}(\alpha'_1-\alpha'_2)}{\beta_{11}\beta_{44}-\beta_{14}^2-\left(\beta_{22}\beta_{44}-\beta_{24}^2\right)}, \tag{17}$$

and

$$g_1 = \frac{\beta_{14}+\beta_{24}}{\beta_{44}}, \quad g_{-1} = \frac{\beta_{14}-\beta_{24}}{\beta_{44}}, \quad g_k = \frac{\beta_{14}+k\beta_{24}}{\beta_{44}}, \quad g_{-k} = \frac{\beta_{14}-k\beta_{24}}{\beta_{44}} \tag{18}$$

Back to the solutions of Eq. (15), the stress functions F and Ψ need to be resolved first. The corresponding stress fields σ_r, σ_ϕ and $\sigma_{\phi z}$ are found from Eqs. (10) and (11) as

$$\sigma_r = \eta_1\sigma_{r\phi}^0/r^2 + c_3/r + 2c_5 + (1-k)c_6 r^{-k-1} + (1+k)c_7 r^{k-1} + \vartheta\mu_1 r,$$

$$\sigma_\phi = -\eta_1\sigma_{r\phi}^0/r^2 + 2c_5 - k(1-k)c_6 r^{-k-1} + k(k+1)c_7 r^{k-1} + 2\vartheta\mu_1 r,$$

$$\sigma_{\phi z} = -\eta_2\sigma^0_{r\phi}/r^2 - \left(\beta_{11}c_3 + \beta_{15}\sigma^0_{rz}\right)/\left(\beta_{14}r\right) - \left[2g_1c_5 + s'_{34}/\beta_{44}\varepsilon^0_z + \alpha'_4/\beta_{44}\Delta T\right] - (1 - k)g_{-k}c_6r^{-k-1} - (1+k)g_kc_7r^{k-1} + \vartheta\mu_2 r, \tag{19}$$

together with the shearing stresses (4) and axial stress (8), where c_i being some constants to be determined. However, in certain geometric configurations the boundary conditions together with the end conditions alone are not sufficient for the determination of all the unknown coefficients (Timoshenko and Goodier, 1970). In particular, for a complete ring additional investigations are necessary to ensure the displacements are single-valued. To do this, by integration of Eq. (14a), we obtain

$$U = \int\left(\beta_{11}\sigma_r + \beta_{12}\sigma_\phi + \beta_{14}\sigma_{\phi z} + \beta_{15}\sigma_{rz} + \beta_{16}\sigma_{r\phi} + s'_{13}\varepsilon^0_z + \alpha'_1\Delta T\right)\mathrm{d}r + p(\phi) + p_0, \tag{20}$$

in which $p(\phi)$ is a periodic function of ϕ with a period 2π, and p_0 is a constant. From Eqs. (20) and (14b), we find

$$\begin{aligned}\frac{\partial V}{\partial\phi} &= \left[\left(\beta_{12}\beta_{14} - \beta_{11}\beta_{24}\right)/\beta_{14}c_3 - \left(\beta_{15}\beta_{24} - \beta_{14}\beta_{15}\right)/\beta_{14}\sigma^0_{rz} - p_0\right] + \Big\{\left[s'_{23} - s'_{13} + s'_{34}\left(\beta_{14}\right.\right.\\ &\quad \left.\left. - \beta_{24}\right)/\beta_{44}\right]\varepsilon^0_z + 2\left[\beta_{22} - \beta_{11} + g_1\left(\beta_{14} - \beta_{24}\right)\right] + c_5\left[\alpha'_2 - \alpha'_1 + \alpha'_4\left(\beta_{14} - \beta_{24}\right)/\beta_{44}\right]\Delta T\Big\}r\\ &\quad + \left[\left(\beta_{22} - \beta_{11}\right)\eta_1 + \left(\beta_{14} + \beta_{24}\right)\eta_2 + \beta_{26}\right]\sigma^0_{r\phi}/r + \left[\left(4\beta_{22} - \beta_{11}\right)\mu_1 + \left(2\beta_{24} - \beta_{14}\right)\mu_2\right]\vartheta r^2/2\\ &\quad + \frac{1-k}{k}\left[\beta_{11} - \beta_{22}k^2 - g_{-k}\left(\beta_{14} + \beta_{24}k^2\right)\right]c_6r^{-k}\\ &\quad - \frac{1+k}{k}\left[\beta_{11} - \beta_{22}k^2 - g_k\left(\beta_{14} - \beta_{24}k^2\right)\right]c_7r^k - p(\phi).\end{aligned} \tag{21}$$

Surprisingly after some derivations it can be shown that the coefficients of the terms r^{-1}, r^2, r^k, r^{-k} turn out to be zero identically. For V to be single-valued, it is necessary that $\partial V/\partial\phi$ must at most be a periodic function of ϕ with period 2π. Thus the coefficients of the r and constant terms should be set zero. This provides

$$\left(\beta_{12}\beta_{14} - \beta_{11}\beta_{24}\right)/\beta_{14}c_3 - \left(\beta_{15}\beta_{24} - \beta_{14}\beta_{15}\right)/\beta_{14}\sigma^0_{rz} - p_0 = 0, \tag{22a}$$

$$c_5 = \frac{\xi_\varepsilon\varepsilon^0_z + \xi_t\Delta T}{2}. \tag{22b}$$

Next from Eq. (14e), by direct expansions, we obtain

$$\begin{aligned}\frac{\partial W}{\partial\phi} &= \left[\left(\beta^2_{14} - \beta_{11}\beta_{44}\right)c_3 - \left(\beta_{15}\beta_{44} - \beta_{14}\beta_{45}\right)\sigma^0_{rz}\right]/\beta_{14} + \left[\left(\beta_{14} - \beta_{24}\right)\eta_1 + \beta_{44}\eta_2 + \beta_{46}\right]\sigma^0_{r\phi}/r\\ &\quad + 2\left(\beta_{14} + \beta_{24} - \beta_{44}g_1\right)c_5r + \left[\left(\beta_{14} + 2\beta_{24}\right)\mu_1 + \beta_{44}\mu_2 - 1\right]\vartheta r^2 + (1-k)\left(\beta_{14} - \beta_{24}k\right.\\ &\quad \left. - \beta_{44}g_{-k}\right)c_6r^{-k} + (1+k)\left(\beta_{14} + \beta_{24}k - \beta_{44}g_k\right)c_7r^k.\end{aligned} \tag{23}$$

Again, interestingly, after expansion and use of Eqs. (17) and (18), the coefficients of the terms r^{-1}, r, r^2, r^{-k} and r^k can be shown to be zero. To ensure the single-valuedness of the displacement the constant term must be set zero, which leads to $c_3 = \rho_1\sigma_{rz}^0$. This together with Eq. (22b) will provide p_0 in terms of σ_{rz}^0.

To find the displacements V and W, we make use of the relations (14c) and (14d). First, integration of Eq. (21) will give

$$V = -\int p(\phi)\,\mathrm{d}\phi + \tilde{V}(r), \tag{24}$$

$\tilde{V}(r)$ being a function of r only. Substituting Eqs. (24) and (20) into Eq. (14c), we find

$$\frac{\mathrm{d}p(\phi)}{\mathrm{d}\phi} + \int p(\phi)\,\mathrm{d}\phi = 0, \tag{25a}$$

,

$$r\frac{\mathrm{d}V(r)}{\mathrm{d}r} - \tilde{V}(r) = r\Big(\beta_{16}\sigma_r + \beta_{26}\sigma_\phi + \beta_{46}\sigma_{\phi z} + \beta_{56}\sigma_{rz} + \beta_{66}\sigma_{r\phi} + s'_{36}\varepsilon_z^0 + \alpha'_6\Delta T\Big) \tag{25b}$$

from which $p(\phi) = a\cos\phi + b\sin\phi$ and the homogeneous solution of $\tilde{V}(r)$ is $\tilde{a}r$, both of which can be incorporated into rigid body motions. By expansion, the nonhomogeneous term (right-hand side) of Eq. (25b) is

$$\begin{aligned}&\Big[(\beta_{14}\beta_{16} - \beta_{11}\beta_{46})c_3 - (\beta_{15}\beta_{46} - \beta_{14}\beta_{56})\sigma_{rz}^0\Big]/\beta_{14} + \big[(\beta_{16} - \beta_{26})\eta_1 + \beta_{46}\eta_2 + \beta_{66}\big]\sigma_{r\phi}^0/r + \Big[(\beta_{16}\\ &\quad + \beta_{26} - \beta_{46}g_1)\xi_\varepsilon + (s'_{36} - \beta_{46}s'_{34}/\beta_{44})\Big]r\varepsilon_z^0 + \Big[(\beta_{16} + \beta_{26} - \beta_{46}g_1)\xi_t + (\alpha'_6 - \beta_{46}\alpha'_4/\beta_{44})\Big]r\Delta T\\ &\quad + \big[(\beta_{16} + 2\beta_{26})\mu_1 + \beta_{46}\mu_2\big]\vartheta r^2 + (1-k)(\beta_{16} - \beta_{26}k - \beta_{46}g_{-k})c_6r^{-k} + (1+k)(\beta_{16} + \beta_{26}k\\ &\quad - \beta_{46}g_k)c_7r^k.\end{aligned} \tag{26}$$

Thus, the displacement V is exactly the particular solution $\tilde{V}_p(r)$ of Eq. (25b), which can be easily solved by the method of undetermined coefficients. Lastly, the displacement W is found via integration of Eq. (14d).

In summary, the final forms of admissible stress fields are expressed by

$$\begin{aligned}\sigma_r &= \eta_1\sigma_{r\phi}^0/r^2 + \rho_1\sigma_{rz}^0/r + \xi_\varepsilon\varepsilon_z^0 + \xi_t\Delta T + (1-k)c_6r^{-k-1} + (1+k)c_7r^{k-1} + \vartheta\mu_1 r,\\ \sigma_\phi &= -\eta_1\sigma_{r\phi}^0/r^2 + \xi_\varepsilon\varepsilon_z^0 + \xi_t\Delta T - k(1-k)c_6r^{-k-1} + k(k+1)c_7r^{k-1} + 2\vartheta\mu_1 r,\\ \sigma_{\phi z} &= -\eta_2\sigma_{r\phi}^0/r^2 - \rho_2\sigma_{rz}^0/r - (\xi_\varepsilon g_1 + s'_{34}/\beta_{44})\varepsilon_z^0 - (\xi_t g_1 + \alpha'_4/\beta_{44})\Delta T - (1-k)g_{-k}c_6r^{-k-1}\\ &\quad - (1+k)g_kc_7r^{k-1} + \vartheta\mu_2 r,\end{aligned} \tag{27}$$

together with in-plane stress $\sigma_{r\phi}$ and anti-plane stress σ_{rz} given in Eq. (4) and axial stress in Eq. (8), where the coefficients c_6, c_7, $\sigma_{r\phi}^0$, σ_{rz}^0, ε_z^0 and ϑ are unknown constants to be determined from the boundary conditions on the lateral surfaces and at the end sections.

The admissible displacement fields are

$$
\begin{aligned}
U = {} & \kappa_1\sigma_{rz}^0 - \left[(\beta_{11}-\beta_{12})\eta_1 + \beta_{14}\eta_2 + \beta_{16}\right]\sigma_{r\phi}^0/r + \left[(\beta_{11}+2\beta_{12})\mu_1 + \beta_{14}\mu_2\right]\vartheta r^2/2 + \Big[(\beta_{11}+\beta_{12} \\
& - \beta_{14}g_1)\xi_\varepsilon + \left(s'_{13} - \beta_{14}s'_{34}/\beta_{44}\right)\Big]r\varepsilon_z^0 + \Big[(\beta_{11}+\beta_{12}-\beta_{14}g_1)\xi_t + \left(\alpha'_1 - \beta_{14}\alpha'_3/\beta_{44}\right)\Big]r\Delta T \\
& - \frac{1-k}{k}\left(\beta_{11} - \beta_{12}k - \beta_{14}g_{-k}\right)c_6r^{-k} + \frac{1+k}{k}\left(\beta_{11}+\beta_{12}k - \beta_{14}g_k\right)c_7r^k,
\end{aligned}
\tag{28}
$$

$$
\begin{aligned}
V = {} & -\kappa_2\sigma_{rz}^0 - \frac{1}{2}\left[(\beta_{16}-\beta_{26})\eta_1 + \beta_{46}\eta_2 + \beta_{66}\right]\sigma_{r\phi}^0/r + \left[(\beta_{16}+2\beta_{26})\mu_1 + \beta_{46}\mu_2\right]\vartheta r^2 + \Big[(\beta_{16}+\beta_{26} \\
& - \beta_{46}g_1)\xi_\varepsilon + \left(s'_{36} - \beta_{46}s'_{34}/\beta_{44}\right)\Big]r\varepsilon_z^0 + \Big[(\beta_{16}+\beta_{26}-\beta_{46}g_1)\xi_t + \left(\alpha'_6 - \beta_{46}\alpha'_4/\beta_{44}\right)\Big]r\Delta T \\
& - \frac{1-k}{1+k}\left(\beta_{16} - \beta_{26}k - \beta_{46}g_{-k}\right)c_6r^{-k} - \frac{1+k}{1-k}\left(\beta_{16}+\beta_{26}k - \beta_{46}g_k\right)c_7r^k,
\end{aligned}
\tag{29}
$$

$$
\begin{aligned}
W = {} & \kappa_3\sigma_{rz}^0\ln r - \left[(\beta_{15}-\beta_{25})\eta_1 + \beta_{45}\eta_2 + \beta_{56}\right]\sigma_{r\phi}^0/r + \left[(\beta_{15}+2\beta_{25})\mu_1 + \beta_{45}\mu_2\right]\vartheta r^2/2 \\
& + \Big[(\beta_{15}+\beta_{25}-\beta_{45}g_1)\xi_\varepsilon + \left(s'_{35} - \beta_{45}s'_{34}/\beta_{44}\right)\Big]r\varepsilon_z^0 + \Big[(\beta_{15}+\beta_{25}-\beta_{45}g_1)\xi_t \\
& + \left(\alpha'_5 - \beta_{45}\alpha'_4/\beta_{44}\right)\Big]r\Delta T - \frac{1-k}{k}\left(\beta_{15} - \beta_{25}k - \beta_{45}g_{-k}\right)c_6r^{-k} \\
& + \frac{1+k}{k}\left(\beta_{15}+\beta_{25}k - \beta_{45}g_k\right)c_7r^k,
\end{aligned}
\tag{30}
$$

where

$$
\begin{aligned}
\kappa_1 &= \left(\frac{\beta_{14}\beta_{12}}{\beta_{11}} - \beta_{24}\right)\frac{\beta_{14}\beta_{15}-\beta_{11}\beta_{45}}{\beta_{14}^2-\beta_{11}\beta_{44}} + \left(\beta_{25} - \frac{\beta_{15}\beta_{12}}{\beta_{11}}\right), \\
\kappa_2 &= \left(\frac{\beta_{14}\beta_{16}}{\beta_{11}} - \beta_{46}\right)\frac{\beta_{14}\beta_{15}-\beta_{11}\beta_{45}}{\beta_{14}^2-\beta_{11}\beta_{44}} + \left(\beta_{56} - \frac{\beta_{15}\beta_{16}}{\beta_{11}}\right), \\
\kappa_3 &= \left(\frac{\beta_{14}\beta_{15}}{\beta_{11}} - \beta_{45}\right)\frac{\beta_{14}\beta_{15}-\beta_{11}\beta_{45}}{\beta_{14}^2-\beta_{11}\beta_{44}} + \left(\beta_{55} - \frac{\beta_{15}^2}{\beta_{11}}\right).
\end{aligned}
\tag{31}
$$

3.2. Anisotropic materials with $k = 2$

From the positive definiteness of the elastic compliance it is seen that the value of k (Eq. (16)) may take any positive quantity. For materials with $k = 2$, the non-homogeneous term of Eq. (15) coincides with one of the homogeneous solutions, and thus, in contrast to those in Section 3.1, the particular solutions (15) need to be carefully modified. However, in practical situations it would appear unlikely that such coincidence, $k = 2$, would occur, since all material coefficients are measured quantities. Detailed solutions of this kind are of little practical use. For brevity only the stress potentials are solved:

$$F = c_1 + \eta_1 \sigma^0_{r\phi} \ln r + c_3 r + c_5 r^2 + c_6/r + c_7 r^3 + \lambda_1 \vartheta r^3 \ln r,$$

$$\Psi = c_1' + \eta_2 \sigma^0_{r\phi} \ln r + \left(\beta_{11} c_3 + \beta_{15}\sigma^0_{rz}\right)/\beta_{14} r \ln r - \frac{1}{2} g_{-2} c_6/r + \frac{1}{2} g_2 \lambda \vartheta r^3 \ln r + \left[g_1 c_5 + \frac{1}{2\beta_{44}}\left(s_{34}'\varepsilon^0_z + \alpha_4' \Delta T\right)\right] r^2 - \left(\frac{2 + 3\beta_{14}\lambda}{12\beta_{44}}\vartheta - \frac{1}{2} g_2 c_7\right) r^3, \tag{32}$$

where

$$\lambda = \frac{2\beta_{24} - \beta_{14}}{3\left(\beta_{14}^2 - \beta_{11}\beta_{44}\right)}. \tag{33}$$

Again, the stress fields are derived from (10), (11) and (4), and the displacements can be integrated from the elastic strains taking into account of global compatibility to ensure the single-valuedness around each contour of a multiply-connected domain.

3.3. Anisotropic materials with $k = 1$

As remarked earlier, a transversely isotropic solid and its higher symmetric classes correspond to $k = 1$. In addition to these, theoretically there are situations that k equals to 1, namely $\beta_{44}\beta_{11} - \beta_{14}^2 = \beta_{44}\beta_{22} - \beta_{24}^2$, but the material is fully anisotropic. In this particular coincidence, '0' is a characteristic root of multiple three, and '1' and '2' are roots of multiple two, and the solutions for the potentials need to be revised. As the same reasoning for $k = 2$, there are very few occasions that this coincidence would occur. Only solutions of the potentials are recorded here:

$$F = c_1 + c_2 \ln r + c_3 r + c_5 r^2 + c_6 \ln^2 r + c_7 r^2 \ln r + \mu_1 \vartheta r^3/3,$$

$$\Psi = c_1' + \left[g_{-1} c_2 + \left(2\beta_{14} c_6 + \beta_{46}\sigma^0_{r\phi}\right)/\beta_{44}\right] \ln r + \frac{\beta_{11} c_3 + \beta_{15}\sigma^0_{rz}}{\beta_{14}} r \ln r - \frac{1}{6}\mu_2 r^3 \vartheta + c_3' r + \left[g_1 c_5 - \frac{\beta_{14}}{\beta_{44}} c_7 + \frac{1}{2\beta_{44}}\left(s_{34}'\varepsilon^0_z + \alpha_4'\right)\Delta T\right] r^2 - g_{-1} c_6 \ln^2 r + g_1 c_7 r^2 \ln r, \tag{34}$$

where c_i are constants to be determined from boundary conditions.

3.4. Monoclinic and orthotropic materials

Suppose that the material is monoclinic, namely possessing a reflectional symmetry at $z = 0$. It is known that (Nye, 1957)

$$\beta_{14} = \beta_{24} = \beta_{15} = \beta_{25} = \beta_{46} = \beta_{56} = 0,$$

$$s_{34}' = s_{35}' = \alpha_4' = \alpha_5' = 0. \tag{35}$$

In this case the governing equations for the potentials F and Ψ in Eq. (12) will become totally independent

$$\beta_{22}\left(\frac{\mathrm{d}^4F}{\mathrm{d}r^4}+\frac{2}{r}\frac{\mathrm{d}^3F}{\mathrm{d}r^3}\right)+\beta_{11}\left(-\frac{1}{r^2}\frac{\mathrm{d}^2F}{\mathrm{d}r^2}+\frac{1}{r^3}\frac{\mathrm{d}F}{\mathrm{d}r}\right)=-2\left(\beta_{16}+\beta_{26}\right)\sigma^0_{r\phi}/r^4,$$

$$\beta_{44}\left(\frac{\mathrm{d}^3\Psi}{\mathrm{d}r^3}+\frac{1}{r}\frac{\mathrm{d}^2\Psi}{\mathrm{d}r^2}\right)=-2\vartheta. \tag{36}$$

This implies that the stresses σ_r, σ_ϕ, and $\sigma^0_{r\phi}$ have no linkage with $\sigma^0_{z\phi}$. Again to ensure the displacement single-valued for a multiply-connected domain, one needs to integrate the displacements (14) properly. In summary, the admissible fields of the stresses and displacements are

$$\begin{cases}\sigma_r=\left(\xi_\varepsilon\varepsilon^0_z+\xi_t\Delta T\right)+(1-k)c_1r^{-k-1}+(1+k)c_2r^{k-1}-\dfrac{\beta_{16}+\beta_{26}}{\beta_{11}-\beta_{22}}\dfrac{\sigma^0_{r\phi}}{r^2},\\ \sigma_\phi=\left(\xi_\varepsilon\varepsilon^0_z+\xi_t\Delta T\right)-k(1-k)c_1r^{-k-1}+k(1+k)c_2r^{k-1}+\dfrac{\beta_{16}+\beta_{26}}{\beta_{11}-\beta_{22}}\dfrac{\sigma^0_{r\phi}}{r^2},\end{cases} \tag{37}$$

$$\sigma_{\phi z}=-\frac{\beta_{45}}{\beta_{44}}\frac{\sigma^0_{rz}}{r}+\frac{\vartheta r}{\beta_{44}}, \tag{38}$$

$$U=\left[\left(\beta_{11}+\beta_{12}\right)\left(\xi_\varepsilon\varepsilon^0_z+\xi_t\Delta T\right)+s'_{13}\varepsilon^0_z+\alpha'_1\Delta T\right]r+\frac{1-k}{k}\left(\beta_{12}k-\beta_{11}\right)c_1r^{-k}+\frac{1+k}{k}\left(\beta_{12}k+\beta_{11}\right)c_2r^k$$
$$+\left[\left(\beta_{11}-\beta_{12}\right)\frac{\beta_{11}+\beta_{26}}{\beta_{11}-\beta_{22}}-\beta_{16}\right]\sigma^0_{r\phi}/r,$$

$$V=\left[\left(\beta_{16}+\beta_{26}\right)\left(\xi_\varepsilon\varepsilon^0_z+\xi_t\Delta T\right)+s'_{36}\varepsilon^0_z+\alpha'_6\Delta T\right]r\ln r-\frac{1-k}{1+k}\left(\beta_{16}-\beta_{26}k\right)c_1r^{-k}-\frac{1+k}{1-k}\left(\beta_{26}k\right.$$
$$\left.+\beta_{16}\right)c_2r^k-\frac{1}{2}\left[\frac{\beta^2_{16}-\beta^2_{26}}{\beta_{11}-\beta_{22}}+\beta_{66}\right]\sigma^0_{r\phi}/r,$$

$$W=\left(\beta_{55}-\frac{\beta^2_{45}}{\beta_{44}}\right)\sigma^0_{rz}\ln r+\frac{\beta_{45}}{2\beta_{44}}\vartheta r^2, \tag{39}$$

where the quantities ξ_t, ξ_ε and ξ_t follow the reduced forms

$$k=\sqrt{\beta_{11}/\beta_{22}},\quad \xi_\varepsilon=\frac{s'_{13}-s'_{23}}{\beta_{22}-\beta_{11}},\quad \xi_t=\frac{\alpha'_1-\alpha'_2}{\beta_{22}-\beta_{11}}. \tag{40}$$

If the material possesses further reflectional symmetries with respect to the $r-z$ plane and $\phi-z$ plane, the material is orthotropic. In this situation, in addition to Eq. (35), the coefficients follow

$$\beta_{16}=\beta_{26}=\beta_{45}=s'_{36}=\alpha'_6=0. \tag{41}$$

The admissible fields readily follow the reduced forms of Eqs. (37) and (39) as

$$\sigma_r=\left(\xi_\varepsilon\varepsilon^0_z+\xi_t\Delta T\right)+(1-k)c_1r^{-k-1}+(1+k)c_2r^{k-1},$$

$$\sigma_\phi = \left(\xi_\varepsilon \varepsilon_z^0 + \xi_t \Delta T\right) - k(1-k)c_1 r^{-k-1} + k(1+k)c_2 r^{k-1},$$

$$\sigma_{\phi z} = \frac{\vartheta r}{\beta_{44}}, \quad \sigma_{r\phi} = \frac{\sigma_{r\phi}^0}{r^2}, \quad \sigma_{rz} = \frac{\sigma_{rz}^0}{r}; \tag{42}$$

$$U = \left[\left(\beta_{11} + \beta_{12}\right)\left(\xi_\varepsilon \varepsilon_z^0 + \xi_t \Delta T\right) + s_{13}' \varepsilon_z^0 + \alpha_1' \Delta T\right] r$$

$$+ \frac{1-k}{k}\left(\beta_{12}k - \beta_{11}\right)c_1 r^{-k} + \frac{1+k}{k}\left(\beta_{12}k + \beta_{11}\right)c_2 r^k,$$

$$V = -\frac{1}{2}\beta_{66}\sigma_{r\phi}^0 / r,$$

$$W = \beta_{55}\sigma_{rz}^0 \ln r. \tag{43}$$

Equivalent but not formally identical results, which are in terms of stiffness, were given by, see for example, Chen et al. (1990), Hashin (1990) and Christensen (1994).

3.5. Transversely isotropic materials

When the material possesses further symmetry such that the transverse plane becomes isotropic, there will be no distinction between cylindrical anisotropy and rectilinear anisotropy. All previous results will reduce to classical results in the literature. In transversely isotropic solids, besides Eqs. (35) and (41), the coefficients are connected by

$$\beta_{11} = \beta_{22}, \quad \beta_{44} = \beta_{55}, \quad \beta_{66} = 2\left(\beta_{11} - \beta_{12}\right), \quad s_{13}' = s_{23}', \quad \alpha_1' = \alpha_2', \tag{44}$$

so that k in Eq. (16) is identically equal to one. The admissible fields reduce to the known results (cf. Timoshenko and Goodier, 1970)

$$\sigma_r = 2c_1 + \frac{c_2}{r^2}, \quad \sigma_\phi = 2c_1 - \frac{c_2}{r^2}, \quad \sigma_{\phi z} = \frac{\vartheta r}{\beta_{44}}, \tag{45}$$

$$U = \left(\beta_{12} - \beta_{11}\right)c_2 / r + \left[2\left(\beta_{11} + \beta_{12}\right)c_1 + s_{13}' \varepsilon_z^0 + \alpha_1' \Delta T\right] r,$$

$$V = -\frac{1}{2}\beta_{66}\sigma_{r\phi}^0 / r,$$

$$W = \beta_{55}\sigma_{rz}^0 \ln r. \tag{46}$$

Again, the constants c_1, c_2, ε_z^0, $\sigma_{r\phi}^0$, σ_{rz}^0 and ϑ will determined from boundary conditions.

4. Circular tube subjected to various loadings

Consider a circular tube of cylindrical anisotropy with inner radius a and outer radius b. Suppose the tube is sufficiently long in the axial direction. We examine the field solutions of the tube subjected to various loadings. Specifically, these include a radial traction, an in-plane shearing stress, and an anti-plane shearing stress on the lateral surfaces together with an axial load P and a twisting moment T applied at the end sections. The latter conditions are equivalent to

$$\int_0^{2\pi}\int_a^b \sigma_z r \,\mathrm{d}r\,\mathrm{d}\phi = P, \tag{47a}$$

$$\int_0^{2\pi}\int_a^b \sigma_{\phi z} r^2 \,\mathrm{d}r\,\mathrm{d}\phi = T. \tag{47b}$$

Alternatively, the latter conditions may be replaced, respectively, by prescribing a constant axial strain ε_z^0 and a twisting angle ϑ (per unit length along the tube).

4.1. Circular tube subjected to radial traction

Consider a circular tube subjected to radial stresses σ_{i} and σ_{o}, at the inner and outer surfaces, respectively. The boundary conditions are

$$\begin{aligned} &\sigma_r = \sigma_{\mathrm{i}}, \quad \sigma_{r\phi} = \sigma_{rz} = 0, \quad \text{at } r = a, \\ &\sigma_r = \sigma_{\mathrm{o}}, \quad \sigma_{r\phi} = \sigma_{rz} = 0, \quad \text{at } r = b, \end{aligned} \tag{48}$$

on the lateral surfaces, and $P = T = 0$ at the end sections.

We first consider that the tube is made of a cylindrically anisotropic material in which $k \neq 1$ and $k \neq 2$. Upon substituting Eqs. (27) and (4) into Eq. (48) readily provides $\sigma_{r\phi}^0 = 0$, $\sigma_{rz}^0 = 0$, and

$$\sigma_{\mathrm{i}} = \xi_\varepsilon \varepsilon_z^0 + \xi_t \Delta T + (1-k)c_1 a^{-k-1} + (1+k)c_2 a^{k-1} + \vartheta\mu_1 a, \tag{49a}$$

$$\sigma_{\mathrm{o}} = \xi_\varepsilon \varepsilon_z^0 + \xi_t \Delta T + (1-k)c_1 b^{-k-1} + (1+k)c_2 b^{k-1} + \vartheta\mu_1 b, \tag{49b}$$

$$P = 0 = \frac{2\pi}{s_{33}}\int_a^b \Big[\varepsilon_z^0 - \big(a_{13}\sigma_r + a_{23}\sigma_\phi + a_{34}\sigma_{\phi z} + \alpha_3\Delta T\big)\Big] r\,\mathrm{d}r, \tag{49c}$$

$$\begin{aligned} T = 0 = -2\pi\int_a^b \Big\{&\big(\xi_\varepsilon g_1 + s_{34}'/\beta_{44}\big)\varepsilon_z^0 + \big(\xi_t g_1 + \alpha_4'/\beta_{44}\big)\Delta T + (1-k)g_{-k}c_1 r^{-k-1} + (1+k)g_k c_2 r^{k-1} \\ &- \vartheta\mu_2 r\Big\} r^2\,\mathrm{d}r. \end{aligned} \tag{49d}$$

which will suffice to determine the four unknown constants c_1, c_2, ε_z^0 and ϑ.

On the other hand, if the end conditions are the prescribed quantities ε_z^0 and ϑ instead of $P = T = 0$ at the ends, then only two knowns, c_1, c_2, need to be determined. In this case, the right-hand-side of Eqs. (49c) and (49d) are the induced axial force (reaction force) and the twisting moment at the end sections which could be actually evaluated. In any event, the nonvanishing stress components are σ_r, σ_ϕ, σ_z, $\sigma_{\phi z}$

Suppose the material possesses at least a symmetry plane at $z = 0$, i.e. monoclinic symmetry, $\sigma_{\phi z}$ will be independent from σ_r and σ_ϕ. In this situation, the system (49) is changed to

$$\sigma_{\mathrm{i}} = \xi_\varepsilon \varepsilon_z^0 + \xi_t \Delta T + (1-k)c_1 a^{-k-1} + (1+k)c_2 a^{k-1}, \tag{50a}$$

$$\sigma_{\mathrm{o}} = \xi_\varepsilon \varepsilon_z^0 + \xi_t \Delta T + (1-k)c_1 b^{-k-1} + (1+k)c_2 b^{k-1}, \tag{50b}$$

$$P = 0 = \frac{2\pi}{s_{33}} \int_a^b \Big[\varepsilon_z^0 - \big(a_{13}\sigma_r + a_{23}\sigma_\phi + \alpha_3 \Delta T\big)\Big] r\,\mathrm{d}r. \tag{50c}$$

This allows one to determine the unknown coefficients c_1, c_2 and ε_z^0. Again, if the end force condition is replaced with a given axial strain ε_z^0, Eqs. (50a) and (50b) are sufficient for determining the unknowns c_1, c_2, while Eq. (50c) is the induced axial equivalent force. The nonvanishing stresses in materials of this kind are σ_r, σ_ϕ and σ_z.

4.2. Circular tube subjected to in-plane shear

Consider that the tube is subjected to in-plane shear at the inner and outer surfaces. The boundary conditions is written as

$$\sigma_{r\phi} = \frac{\tau}{a^2}, \quad \sigma_r = \sigma_{rz} = 0, \quad \text{at } r = a,$$

$$\sigma_{r\phi} = \frac{\tau}{b^2}, \quad \sigma_r = \sigma_{rz} = 0, \quad \text{at } r = b, \tag{51}$$

together with forces free conditions at the ends. This readily provides $\sigma_{r\phi}^0 = \tau$, $\sigma_{rz}^0 = 0$ and the set of equations

$$\sigma_{\mathrm{i}} = \eta_1 \tau / a^2 + \xi_\varepsilon \varepsilon_z^0 + \xi_t \Delta T + (1-k)c_1 a^{-k-1} + (1+k)c_2 a^{k-1} + \vartheta \mu_1 a, \tag{52a}$$

$$\sigma_{\mathrm{o}} = \eta_1 \tau / b^2 + \xi_\varepsilon \varepsilon_z^0 + \xi_t \Delta T + (1-k)c_1 b^{-k-1} + (1+k)c_2 b^{k-1} + \vartheta \mu_1 b, \tag{52b}$$

$$P = 0 = \frac{2\pi}{s_{33}} \int_a^b \Big[\varepsilon_z^0 - \big(a_{13}\sigma_r + a_{23}\sigma_\phi + a_{34}\sigma_{\phi z} + a_{36}\sigma_{r\phi} + \alpha_3 \Delta T\big)\Big] r\,\mathrm{d}r, \tag{52c}$$

$$\begin{aligned} T = 0 = -2\pi \int_a^b \{ &\big(\xi_\varepsilon g_1 + s'_{34}/\beta_{44}\big)\varepsilon_z^0 + \big(\xi_t g_1 + \alpha'_4/\beta_{44}\big)\Delta T \\ &+ (1-k)g_{-k} c_1 r^{-k-1} + (1+k) g_k c_2 r^{k-1} - \vartheta \mu_2 r - \eta_2 \tau / r^2 \} r^2\,\mathrm{d}r. \end{aligned} \tag{52d}$$

for determining the four unknown coefficients c_1, c_2, ε_z^0, ϑ. Again, if ε_z^0 and ϑ instead of P and T are prescribed, Eqs. (52a) and (52b) will suffice to determine c_1, c_2, while Eqs. (52c) and (52d) give the reaction forces at the ends. Except for σ_{rz}, all stresses are non-zero.

For materials with monoclinic symmetry or its higher symmetric classes, $\sigma_{\phi z}$ and ϑ, which are independent of other stress components, are identically zero in the present loading case.

4.3. Circular tube subjected to anti-plane shear

In loading of this type the boundary conditions are

$$\sigma_{rz} = \frac{\tau}{a}, \quad \sigma_r = \sigma_{r\phi} = 0, \quad \text{at } r = a,$$

$$\sigma_{rz} = \frac{\tau}{b}, \quad \sigma_r = \sigma_{r\phi} = 0, \quad \text{at } r = b. \tag{53}$$

Similar to the steps in previous sections, one can construct four equations for the four unknowns to determine the full fields. For general anisotropic materials, except for $\sigma_{r\phi}$ all stresses are nonzero. For monoclinic systems, $\sigma_{r\phi} = \sigma_{\phi z} = 0$ in the absence of an applied torque.

4.4. Torsion of the circular tube

We consider the tube is subjected to a twisting moment T or a twisting angle ϑ per unit length of the tube. The traction-free condition on the lateral surfaces are $\sigma_r = \sigma_{r\phi} = \sigma_{rz} = 0$ at $r = a$ and b. At the end sections, if $T \neq 0$ and $P = 0$, this will provide four equations to determine the unknown coefficients c_1, c_2, ε_z^0 and ϑ. Alternatively, if at the end sections $\varepsilon_z^0 = 0$ and $\vartheta \neq 0$, then there are only two unknowns need to be determined. In either case, the nonvanishing stress components are σ_r, σ_ϕ, σ_z and $\sigma_{\phi z}$.

If the material of the tube is monoclinic or its higher symmetric classes such as orthotropy, transverse isotropy, remarkably, all solutions follow the same form. In particular, it turns out that $\sigma_{\phi z}$ is the only nonzero stress that can be exactly determined from Eq. (47b) as

$$\vartheta = \frac{2\beta_{44}}{b^4 - a^4} T. \tag{54}$$

4.5. Uniform extension of the circular tube

When the tube is subjected to a uniform extension ε_z^0 and is otherwise traction free. The boundary condition on the lateral surfaces are

$$\sigma_r = \sigma_{r\phi} = \sigma_{rz} = 0, \quad \text{at } r = a,$$

$$\sigma_r = \sigma_{r\phi} = \sigma_{rz} = 0, \quad \text{at } r = b, \tag{55}$$

and $\vartheta = 0$ at the end sections. This leads to $\sigma_{r\phi}^0 = \sigma_{rz}^0 = 0$, and two conditions

$$\xi_\varepsilon \varepsilon_z^0 + (1-k)c_1 a^{-k-1} + (1+k)c_2 a^{k-1} = \sigma_\mathrm{i},$$

$$\xi_\varepsilon \varepsilon_z^0 + (1-k)c_1 b^{-k-1} + (1+k)c_2 b^{k-1} = \sigma_\mathrm{o}, \tag{56}$$

for determining the two unknowns c_1, c_2. In general, σ_r, σ_ϕ, σ_z and $\sigma_{\phi z}$ are nonzero. Note that if $P(\neq 0)$ and $T = 0$ are used for the end conditions, there are four unknowns to be determined from four conditions. When the material is monoclinic, the nonzero stresses will be σ_r, σ_ϕ, σ_z.

4.6. Uniform temperature change of the circular tube

Consider the tube is subjected to a uniform temperature change ΔT and is otherwise traction free. We first set $\vartheta = \varepsilon_z^0 = 0$. Again $\sigma_{r\phi}^0 = \sigma_{rz}^0 = 0$, and the boundary conditions provide that

$$\xi_t \Delta T + (1-k)c_1 a^{-k-1} + (1+k)c_2 a^{k-1} = \sigma_{\mathrm{i}},$$

$$\xi_t \Delta T + (1-k)c_1 b^{-k-1} + (1+k)c_2 b^{k-1} = \sigma_{\mathrm{o}} \tag{57}$$

This will provide the full stresses fields for the tube in which four stresses are nonzero. For monoclinic solids, the stress $\sigma_{\phi z}$ will also be zero. Of course, one may set $P = T = 0$ at the ends. This can still be resolved without any difficulty.

An interesting analogy is observed between the loadings of a uniform axial extension and a uniform temperature change by comparing Eqs. (56) and (57). In particular, by setting a linkage

$$s'_{13} \leftrightarrow \alpha'_1, \quad s'_{23} \leftrightarrow \alpha'_2, \quad s'_{34} \leftrightarrow \alpha'_4, \quad s'_{35} \leftrightarrow \alpha'_5, \quad s'_{36} \leftrightarrow \alpha'_6 \tag{58}$$

the two sets of fields are exactly analogous. This property is also observed in rectilinearly anisotropic solids (Chen, 1998).

5. Some remarks

A complete stress and displacement fields of the tube subjected to radial traction, in-plane shear and anti-plane shear at the inner and outer surfaces are explicitly expressed. Solutions are also presented for the loadings of a simple torsion, a uniform extension and a uniform temperature change. We have outlined a straightforward approach to characterize the stress potentials F and Ψ of the considered problem. This permits one to evaluate the stress fields. In a few special or degenerate cases, such as $k = 1$, $k = 2$, or for a solid possessing a higher symmetry than that of a monoclinic solid, the solutions may require a slight modification depending on the homogenous solutions and the non-homogeneous terms of the governing system. A notable implication of the governing system occurs when the material at least possesses a reflectional symmetry at $z = 0$. In this situation, the solutions for F and Ψ become independent. Admissible displacement fields are integrated from the strains with the requirement that the displacements must be single-valued for a multiply-connected domain. In contrast to the works of Ting (1996, 1999) which require superpositions of two or more basic solutions, the present solutions offer complete solutions ready for direct calculations. To check the correctness of the present results, we have made some numerical comparisons with Ting (1996)'s results which were derived based on elastic stiffness. Except for σ_r term in the torsion loading and the displacement in the uniform extension, all numerical results agree well. Both disagreements are probably due to some typographical errors of the formulae. It may also seem plausible that the present solutions can be applicable to a circular solid bar. This issue has been extensively discussed in Ting (1996, 1999) works. We simply remark that at the center of a circular solid bar, the material should not have anisotropic effect, as there is no distinction between radial and circumferential directions. A discussion of this controversial issue will be reported elsewhere.

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